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PHASE CHANGE IN A SEMI-INFINITE SOLID WITH TEMPERATURE DEPENDENT THERMAL PROPERTIES

MURRAY IMBER and P. N. S. HUANG*

Department of Mechanical Engineering, Polytechnic Institute of Brooklyn, Brooklyn, New York 11201, U.S.A.

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NOMENCLATURE

A_1, B_1,	$\rho_m L/\alpha_{1m}$;
c_1, c_2,	specific heat;
D_1, D_2, D_3, D_4,	defined by equation (12);
L,	latent heat;
$q_{1s}(t)$,	surface heat flux;
S,	melting or solidification distance;
t,	time;
T_{2i},	initial temperature;
T_m,	melting temperature;
T_1, T_2,	temperature, $T - T_m$;
V_1, V_2,	transform variable, defined by equation (1) and equation (2);
x,	distance;
Y, Z,	defined by equation (11);
α_1, α_2,	thermal diffusivity;
w, γ,	$\mathrm{d}\alpha_1/\mathrm{d}V_1\|_{x=S}$;
μ, μ_s,	$2V_{1s}/\rho_m L$;
ρ,	density.

Subscripts

m,	evaluated at the phase change front;
s,	evaluated at the surface.

INTRODUCTION

SINCE the solution to a phase change problem is difficult to obtain, it is customary to introduce the simplification that the thermal properties of the liquid and solid states are constant. For modest temperature ranges, this assumption appears to be reasonable; however, when the temperature differences are large, the validity of this approach is questionable. The purpose of the present investigation is to determine quantitatively, the effect of the variable property assumption in a phase change system.

In what follows the heat balance integral method is amended to include phase change problems with temperature dependent thermal properties in a semi-infinite body initially at its fusion temperature. Melting and freezing situations are considered for various types of surface temperature functions, and generalized expressions are derived for the phase change location which, in turn, determines the requisite temperature profiles. For a quantitative evaluation of the effect of the variable property assumption, the semi-infinite body is assumed to be copper since the thermal properties of its liquid and solid states are well documented in [4].

ANALYSIS

For a semi-infinite body initially at its melting or freezing point, $T_m = 0$, and with temperature dependent thermal properties, analysis of the differential system is expedited by the transformation

$$V_1 = \int_0^{T_1} \rho_1 c_1 \, \mathrm{d}T \tag{1}$$

and

$$V_2 = \mp \rho_m L \tag{2}$$

where the negative and positive designations refer to melting or freezing, respectively. Consequently, the heat conduction equation for each phase is transformed into:

$$\frac{\partial}{\partial x}\left[\alpha_1(V_1)\frac{\partial V_1}{\partial x}\right] = \frac{\partial V_1}{\partial t} \qquad 0 \leqslant x \leqslant S \tag{3}$$

$$V_1(0, t) = V_{1s} \tag{4}$$

$$V_2(x, t) = \mp \rho_m L \qquad S \leqslant x \leqslant \infty \tag{5}$$

$$V_1 = V_{1m} = 0 \text{ at } x = S \tag{6}$$

$$\left.\frac{\partial V_1}{\partial x}\right|_{x=S} = -A_1 \frac{\mathrm{d}S}{\mathrm{d}t} \quad \text{(for melting)} \tag{7a}$$

$$\left.\frac{\partial V_1}{\partial x}\right|_{x=S} = B_1 \frac{\mathrm{d}S}{\mathrm{d}t} \quad \text{(for freezing).} \tag{7b}$$

Henceforth, hyphenated equation numbers are employed to distinguish the melting and freezing processes.

By application of the principles of the heat balance integral method, [1–3], equation (3) is, in turn, reduced to

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[*] Present address: Institute of Nuclear Energy Research, Taiwan, Republic of China.

an ordinary differential equation from which the following expressions at the fusion line are extracted:

$$\alpha_{1m}\frac{\partial^2 V_1}{\partial x^2} + \left(w - \frac{1}{A_1}\right)\left(\frac{\partial V_1}{\partial x}\right)^2 = 0 \quad \text{at } x = S \qquad (8a)$$

$$\alpha_{1m}\frac{\partial^2 V_1}{\partial x^2} + \left(\gamma + \frac{1}{B_1}\right)\left(\frac{\partial V_1}{\partial x}\right)^2 = 0 \quad \text{at } x = S. \qquad (8b)$$

To obtain an explicit expression for the temperature, T_1, a polynomial expansion for the transformed variable V_1, must be chosen. Accordingly, a quadratic expression for the transformed variable V_1 is now assumed,

$$V_1(x, t) = a(x - S) + b(x - S)^2 + c \qquad (9)$$

where the constants a, b and c are readily determined from equations (4), (6) and (8). Hence, equation (9) simplifies to the expression

$$\frac{V_1}{\alpha_{1m}A_1} = \left[\frac{1 - \sqrt{[1 + \mu(1 - wA_1)]}}{(1 - wA_1)}\right]\frac{(x - S)}{S} + \left[\frac{1 - \sqrt{[1 + \mu(1 - wA_1)]}}{(1 - wA_1)} + \frac{\mu}{2}\right]\frac{(x - S)^2}{S^2} \qquad (10a)$$

After performing the required substitutions, the amended form of equation (3) reduces to the non-linear differential equation for the fusion line location

$$\frac{\mathrm{d}}{\mathrm{d}t}(S^2) + \frac{2S^2}{(6 + \mu - Z)}\frac{\mathrm{d}}{\mathrm{d}t}(\mu - Z) = 12\alpha_{1s}\left(\frac{\mu + Z}{6 + \mu - Z}\right) \qquad (11a)$$

with

$$Z = \frac{1 - \sqrt{[1 + \mu(1 - wA_1)]}}{(1 - wA_1)}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}(S^2) + \frac{2S^2}{6 - \mu_s - Y}\frac{\mathrm{d}}{\mathrm{d}t}(\mu_s + Y) = 12\alpha_{1s}\left(\frac{Y - \mu_s}{6 - \mu_s - Y}\right) \qquad (11b)$$

where

$$Y = \frac{1 - \sqrt{[1 - \mu_s(1 + \gamma B_1)]}}{(1 + \gamma B_1)}.$$

Equations (11a) and (11b) are the non-linear differential equations of Bernoulli's type and they have the following solutions:

$$S = \left\{\frac{12\int_0^t \alpha_{1s}\left[\mu^2 + D_1\mu - \frac{\{[\sqrt{1 + \mu(1 - wA_1)}] - 1\}D_2}{(1 - wA_1)}\right]\mathrm{d}t}{6 + \mu + \left[\frac{\sqrt{[1 + \mu(1 - wA_1)]} - 1}{(1 - wA_1)}\right]^2}\right\}^{\frac{1}{2}} \qquad (12a)$$

where

$$D_1 = 6 - \frac{1}{(1 - wA_1)}, \qquad D_2 = 6 - \frac{2}{(1 - wA_1)}$$

and

$$S = \left\{\frac{12\int_0^t \alpha_{1s}\left[\mu_s^2 - D_3\mu_s - D_4\frac{\sqrt{[1 - \mu_s(1 + \gamma B_1)]} - 1}{(1 + \gamma B_1)}\right]\mathrm{d}t}{\left[6 - \mu_s - \frac{1 - \sqrt{[1 - \mu_s(1 + \gamma B_1)]}}{(1 + \gamma B_1)}\right]^2}\right\}^{\frac{1}{2}} \qquad (12b)$$

with

$$D_3 = 6 - \frac{1}{(1 + \gamma B_1)}, \qquad D_4 = 6 - \frac{2}{(1 + \gamma B_1)}.$$

$$\frac{V_1}{\alpha_{1m}B_1} = \left[\frac{1 - \sqrt{[1 - \mu_s(1 + \gamma B_1)]}}{(1 + \gamma B_1)}\right]\frac{(S - x)}{S} + \left[\frac{\mu_s}{2} - \frac{1 - \sqrt{[1 - \mu_s(1 + \gamma B_1)]}}{(1 + \gamma B_1)}\right]\frac{(S - x)^2}{S^2}. \qquad (10b)$$

It should be noted that the terms μ and μ_s shown in the preceding equations have the same algebraic expression, however numerically they are different due to the term, V_{1s}.

MELTING OF A SEMI-INFINITE SOLID

(A) *Constant surface temperature*, T_{1s} = *constant*

When the surface temperature, T_{1s}, is considered to be a constant, μ in turn becomes constant and, equation (12a) simplifies to the general expression

$$S = \left\{ \frac{12\alpha_{1s}\left[\mu + \frac{1 - \sqrt{[1 + \mu(1 - wA_1)]}}{(1 - wA_1)}\right]t}{\left[6 + \mu + \frac{\sqrt{[1 + \mu(1 - wA_1)]}}{(1 - wA_1)}\right]} \right\}^{\frac{1}{2}} \tag{13}$$

In addition, the surface heat flux, q_{1s}, is for any situation

$$q_{1s}(t) = -\alpha_{1s}\frac{\partial V_1}{\partial x}\bigg|_{x=0}$$
$$= \frac{\alpha_{1s}(\rho L)}{S}\left[\mu + \frac{1 - \sqrt{[1 + \mu(1 - wA_1)]}}{(1 - wA_1)}\right]. \tag{14}$$

For comparison purposes, the constant physical property case, corresponds to the situation of, $1 - wA_1 = 1$, where the thermal properties are evaluated at the fusion temperature, T_m. Incidentally, direct substitution of the requirement, $1 - wA_1 = 1$, into equation (13), produces the same results as obtained in [1].

(B) *Linear variation of surface temperature*, $T_{1s} = a_0 + a_1 t$

The physical properties are treated as linear functions of temperature, i.e.

$$\rho_1 c_1 = \rho_m c_{1m}(1 + bT_1), \qquad \alpha_{1s} = \alpha_{1m}(1 + gT_{1s}).$$

In general terms, equation (12a) may be expressed as

$$\frac{S}{2\sqrt{\alpha_{1m}}} = \frac{\sqrt{3}}{\left[6 + \mu + \frac{\sqrt{[1 + \mu(1 - wA_1)]} - 1}{(1 - wA_1)}\right]}$$
$$\times \left\{(E_1 t + E_2 t^2 + E_3 t^3 + E_4 t^4 + E_5 t^5 + E_6 t^6)\right.$$
$$- \frac{D_2}{(1 - wA_1)}\left[H_1(F_0 + F_1 t + F_2 t^2)^{\frac{3}{2}} - [(1 + ga_0)t\right.$$
$$+ \tfrac{1}{2}ga_1 t^2] - H_2(2F_2 t + F_1)\sqrt{(F_0 + F_1 t + F_2 t^2)}$$
$$+ H_3 \sin^{-1}\left(\frac{2F_2 t + F_1}{\sqrt{(F_1^2 - 4F_0F_2)}}\right) - H_1 F_0^{\frac{3}{2}} + H_2 F_1(\sqrt{F_0})$$
$$\left.\left. - H_3 \sin^{-1}\left(\frac{F_1}{\sqrt{(F_1^2 - 4F_0F_2)}}\right)\right]\right\}^{\frac{1}{2}}. \tag{15}$$

The explicit form for the constant terms E_n, F_n and H_n are shown in [5].

FREEZING OF A SEMI-INFINITE LIQUID

(A) *Constant surface temperature*, T_{1s} = *constant*

Accordingly, equations (12b) can be expressed in general form as

$$S = \left\{ \frac{12\alpha_{1s}\left[\frac{1 - \sqrt{[1 - \mu_s(1 + \gamma B_1)]}}{(1 + \gamma B_1)} - \mu_s\right]t}{6 - \mu_s - \frac{1 - \sqrt{[1 - \mu_s(1 + \gamma B_1)]}}{(1 + \gamma B_1)}} \right\}^{\frac{1}{2}}. \tag{16}$$

It should be noted that when, $(1 + \gamma B_1) = 1{\cdot}0$, the results reduce to the same conclusions obtained in [1].

(B) *Linear variation of surface temperature*, $T_{1s} = a_0 + a_1 t$

When the thermophysical equations are successfully approximated by linear function of temperature, i.e.

$$\alpha_{1s} = \alpha_{1m}(1 + gT_{1s}), \qquad \rho_1 c_1 = \rho_m c_{1m}(1 + bT_1)$$

equation (12b) can be expressed in general terms as

$$\frac{S}{2\sqrt{\alpha_{1m}}} = \frac{\sqrt{3}}{6 - \mu_s - \frac{1 - \sqrt{[1 - \mu_s(1 + \gamma B_1)]}}{(1 + \gamma B_1)}}$$
$$\times \left\{(E_1 t + E_2 t^2 + E_3 t^3 + E_4 t^4 + E_5 t^5 + E_6 t^6)\right.$$
$$- \frac{D_4}{(1 + \gamma B_1)}\left[H_1(F_0 + F_1 t + F_2 t^2)^{\frac{3}{2}}\right.$$
$$- [(1 + ga_0)t + \tfrac{1}{2}ga_1 t^2] - H_2(2F_2 t + F_1)\sqrt{(F_0 + F_1 t}$$
$$+ F_2 t^2) + H_3 \sin^{-1}\frac{2F_2 t + F_1}{\sqrt{(F_1 - 4F_0F_2)}} - H_1 F_0^{\frac{3}{2}}$$
$$\left.\left. + H_2 F_1(\sqrt{F_0}) - H_3 \sin^{-1}\frac{F_1}{\sqrt{(F_1^2 - 4F_0F_2)}}\right]\right\}^{\frac{1}{2}}. \tag{17}$$

RESULTS AND CONCLUSIONS

As presented in analysis section, the solution to the nonlinear phase change problem with the linear boundary conditions is obtained via the application of the heat balance integral method. A quadratic expression for the transformed variable $V_1(x, t)$ is assumed, equation (9), and expressions are obtained for the phase change location. The question of increasing the order of the approximating polynomial has been discussed in [2], and it should be noted the conclusion that a higher order polynomial does not necessarily yield a more accurate solution, also applies to the cases considered here. Though not reported in the present investigation, numerical results were obtained for a cubic expression which produced very little difference from those thus far presented. When a polynomial approximation of higher degree than the second is utilized, a parametric representation for $S(t)$ is unobtainable; however the desired information may be computed numerically.

As shown in [5], the fusion line location and the internal temperature were evaluated for a variety of boundary conditions. Figure 1 may be regarded as representative of the results obtained. Quantitatively, it appears that when the thermal properties are computed at the fusion temperature, T_m, the constant property solution is rather close to the variable property solution, for the selected surface temperature variations. This effect is indicated as an intermittent dashed curve for the modified equation (12a), i.e. equation (15).

It seems, therefore, in this one case that the thermal parameters for the variable property case could be approximated

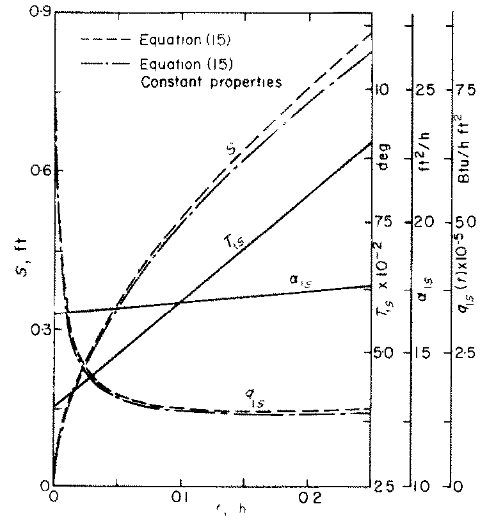

FIG. 1. Melting of copper at $T_{2i} = 0°$ and $T_{1s} = 400 + 2000t$.

rather easily by the simpler constant thermal physical property condition for materials having linear thermal conductivity and volumetric specific heat functional relationships with temperature.

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THE INFLUENCE OF ROTATION ON THE HEAT TRANSFER FROM A SPHERE TO AN AIR STREAM

T. D. EASTOP

The Polytechnic, Wolverhampton, Staffs, England

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NOMENCLATURE

c,	specific heat;
d,	duct diameter [m];
D,	sphere diameter [m];
F,	function defined by equation (4) [W/m² K];
Gr,	Grashof number;
h_c,	heat transfer coefficient for convection [W/m² K];
h_{eff},	effective heat transfer coefficient [W/m² K];
h_R,	heat transfer coefficient for radiation [W/m² K];
k,	thermal conductivity [W/m K];
Nu,	Nusselt number, h_cD/k;
Pr,	Prandtl number;
Q_c,	rate of heat transfer by convection [W];
Q_{cond},	rate of heat transfer by conduction [W];
Q_R,	rate of heat transfer by radiation [W];
R,	radius of the sphere [m];
Re,	Reynolds number, $(U\rho D/\mu)$;
R_{eR},	rotational Reynolds number, VD/ν;
R_{SH},	radius of support shaft [m];
t_a,	air temperature [°C];
t_m,	sphere mean temperature [°C];
t_s,	sphere surface temperature [°C];
T,	absolute temperature of the air in the boundary layer [K];
T_a,	absolute air temperature [K];
T_m,	sphere mean absolute temperature [K];
T_s,	sphere surface absolute temperature [K].